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Efficient algorithms for wavelength assignment on trees of rings[☆]

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ABSTRACT

A fundamental problem in communication networks is wavelength assignment (WA): given a set of routing paths on a network, assign a wavelength to each path such that the paths with the same wavelength are edge-disjoint, using the minimum number of wavelengths. The WA problem is NP-hard for a tree of rings network which is well used in practice. In this paper, we give an efficient algorithm which solves the WA problem on a tree of rings with an arbitrary (node) degree using at most $3L$ wavelengths and achieves an approximation ratio of 2.75 asymptotically, where L is the maximum number of paths on any link in the network. The $3L$ upper bound is tight since there are instances of the WA problem that require $3L$ wavelengths even on a tree of rings with degree four. We also give a $3L$ and 2-approximation (resp. 2.5-approximation) algorithm for the WA problem on a tree of rings with degree at most six (resp. eight). Previous results include: $4L$ (resp. $3L$) wavelengths for trees of rings with arbitrary degrees (resp. degree at most eight), and 2-approximation (resp. 2.5-approximation) algorithm for trees of rings with degree four (resp. six).

1. Introduction

A fundamental problem in communication networks is that given a set of connection requests (source-destination pairs) in a network, find a routing path for each request and assign each path a channel such that the paths with the same channel do not share any communication link in the network. A major optimization problem here is to minimize the number of channels. In wavelength division multiplexing (WDM) optical networks, each channel is supported by a wavelength or a *color*. Given a set P of paths in a network, a *valid coloring* for P is defined to satisfy the property that each path of P is given a single color such that the paths with the same color do not share any common link of the network. The optimization problem for the channel assignment can be expressed as the *routing and wavelength assignment* (RWA) problem: given a set of connection requests, find a path for each request and a valid coloring for the set of found paths, using the minimum number of colors. When the set of paths is pre-defined, the optimization problem is known as the *wavelength assignment* (WA) problem which can be defined as follows: given a set of paths in a network, find a valid coloring for the set of paths, using the minimum number of colors. The RWA and WA problems have been extensively studied in both networks and graph theory (under the term *path coloring*) [9,10,16,19]. Especially, they have been major research topics on WDM optical networks [1,2,6,11,14,17]. In this paper, we study the WA problem on the tree of rings network. Most previous work on trees of rings use the undirected graph as the abstract model of the networks [2,4,7,8,17]. We also use the undirected graph model

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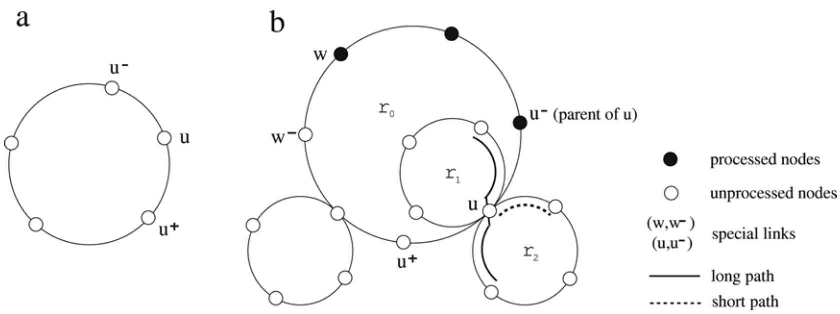

Fig. 1. Illustration of some terms defined on a tree of rings *TR*.

and give efficient algorithms for the WA problem on the undirected trees of rings. In what follows, we consider undirected graphs for networks unless otherwise stated.

A *ring* network is a cycle with at least three nodes. A *tree of rings* can be defined as follows [8]: A single ring is a tree of rings, and the graph obtained by adding a node-disjoint ring R to an existing tree of rings TR and then merging one node of R and one node of TR into one is also a tree of rings (see (b) of Fig. 1 for an example of trees of rings). In a tree of rings, any two rings have at most one node in common, and for any pair (u, v) of nodes in the tree of rings there are exactly two edge-disjoint paths between u and v. A tree of rings is a practical topology, with several sub-rings connected to a main ring, and sub-subrings to the sub-rings, and so on, as observed in [17]. It remains connected even if an arbitrary link fails in each ring, and thus provides a better fault tolerance than a tree network. Many research efforts have been devoted to the study on trees of rings [2,4,5,7,8,14,17].

Both the RWA and WA problems on rings (thus on trees of rings) are NP-hard [9,10]. The approach of *cut-one-link* has been used for the RWA problem on trees of rings. In this approach, one link is removed from each ring in a tree of rings to get a tree network and the solution for the WA problem on the tree is used as the solution of the RWA problem on the tree of rings [14,17]. By this approach, a 3-approximation algorithm is known for the RWA problem on trees of rings [17]. This approach, however, does not work for the WA problem on trees of rings because the paths are pre-defined and may contain the links to be removed.

Given a set of routing paths on a network, let L be the maximum number of paths on any link in the network. Then L is an obvious lower bound on the number of colors for the WA problem on the network. For the WA problem on ring networks, Tucker gave an algorithm which uses at most $2L - 1$ colors and showed that the $2L - 1$ upper bound is tight for the problem [19]. A 1.5-approximation algorithm is known for the WA problem on rings [12]. For the WA problem on trees of rings with (node) degree at most four (i.e. each node can appear in at most two rings), Deng et al. gave a 2-approximation algorithm [7]. For the WA problem on trees of rings with arbitrary degrees, Erlebach [8] proposed an algorithm which uses at most $4L$ colors. For the WA problem on trees of rings with constant degrees, the following results are known [4]: a $3L$ algorithm for degree at most eight; a 2.5-approximation algorithm for degree at most six; and a 2-approximation algorithm for degree at most four. The $3L$ upper bound is tight because it was shown in [4] that there are instances of the WA problem that require $3L$ colors even on a tree of rings with degree four.

In this paper, we give a polynomial time algorithm which uses at most $3L$ colors and achieves an approximation ratio of 2.75 asymptotically for the WA problem on trees of rings with arbitrary degrees. This improves the previous $4L$ and 4-approximation result of [8] and shows that the $3L$ tight upper bound on the number of colors can be achieved for the whole class of trees of rings. Our algorithm is based on a processing order proposed in [8], novel applications of edge-coloring of multigraphs, and efficient path coloring schemes on trees of rings. Based on these techniques, we further give a $3L$ and 2-approximation (resp. 2.5-approximation) algorithm for the WA problem on trees of rings with degree at most six (resp. eight). The algorithms on trees of rings with bounded degrees are of independent interest and improve previous results of [4,7]. Our $3L$ result also implies a 3-approximation algorithm for the RWA problem on trees of rings. The algorithm does not use the cut-one-link approach.

The rest of the paper is organized as follows. Section 2 gives the preliminaries of the paper. In Section 3, we give a simple algorithm which uses at most $3L$ colors for trees of rings with arbitrary degrees and show an instance which requires $3L$ colors. Efficient path coloring schemes for trees of rings and some useful facts on edge-coloring of multigraphs are given in Section 4. We show the 2.75-approximation algorithm (which uses at most $3L$ colors) in Section 5. The algorithms for the WA problem on trees of rings with bounded degrees are presented in Section 6. The final section concludes the paper.

2. Preliminaries

A tree of rings network is denoted by a graph TR with node set $V(TR)$ and link set $E(TR)$. We reserve the terms of *vertex* and *edge* for other graphs used in the paper. The readers may refer to a graph theory book like [3] for basic graph definitions and terminology. We use a *path* for a simple path in TR (i.e. repetition of nodes is not allowed). Two paths in TR *intersect* if they have a common link. Two paths in TR are *edge-disjoint* if they do not intersect. A set of paths in TR is *edge-disjoint* if any two paths in the set do not intersect. We say a path is *on a link* (resp. *a node*) if the path contains the link (resp. the node). We say a path is *on a ring* if the path contains a link of the ring. For TR, we have the following property.

Proposition 1. *For any node $u \in V(TR)$, a path on u can be on at most two rings which contain u.*

For a node u in a ring of TR, we denote u^- as the neighbor of u in the counter-clockwise direction and u^+ as the neighbor of u in the clockwise direction in the ring (see (a) of Fig. 1).

Given a set $W = \{\lambda_1, \lambda_2, \ldots\}$ of colors and a set P of paths, a color assignment from W to P is called a *valid coloring* if each path in P is assigned a single color from W and the paths with the same color are edge-disjoint. Finding a valid coloring for P is also called *coloring* P. Given a set P of paths in TR, let L be the maximum number of paths of P on any link of TR and OPT be the minimum number of colors required for coloring P. Then trivially $L \le OPT$.

Given a (multi)graph G with vertex set $V(G)$ and edge set $E(G)$, an edge-coloring of G is an assignment of colors to the edges of G such that every pair of edges incident to the same vertex are given different colors. We call such an assignment a *valid edge-coloring* of G. We denote the (node) degree of $u \in V(G)$ by $d(u)$ and the degree of G by $\Delta(G) = \max\{d(u) | u \in V(G)\}$. The following results are well known.

Proposition 2 (*[18]*). *A valid edge-coloring of a multigraph G using at most $\lfloor 3\Delta(G)/2 \rfloor$ colors can be found in $O(|E(G)|(\Delta(G) + |V(G)|))$ time.*

Proposition 3 (*[15]*). *A valid edge-coloring of a multigraph G using at most $\max\{\lfloor (11\Delta(G)+8)/10 \rfloor, l(G)\}$ colors can be found in $O(|E(G)|(\Delta(G) + |V(G)|))$ time, where*

$$l(G) = \max\left\{ L(H) = \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil \;\middle|\; H \text{ is a subgraph of } G \text{ with } |V(H)| \ge 3 \right\}$$

is a lower bound on the number of colors for the edge-coloring of G.

A well used strategy for the WA problem is the *first-fit coloring*: Given a set $W = \{\lambda_1, \lambda_2, \ldots\}$ of colors and a set P of paths, the paths in P are colored one by one in arbitrary order, and a path $p \in P$ is assigned a color λ_i with the smallest index i such that no path of $P \setminus \{p\}$ already colored by λ_i intersects with p. We say a set of elements is assigned distinct colors if for any two different elements in the set, the elements are assigned different colors.

Throughout the paper, we will denote W_P as the set of colors assigned to a set P of paths, and denote W_{uv} as the set of colors assigned to the paths on a link (u, v) of TR.

3. 3L upper and lower bounds

In this section, we give a simple algorithm, called A1, which uses at most $3L$ colors for the WA problem on TR with an arbitrary degree and show that the $3L$ upper bound is tight. We first give a framework in Fig. 2 for all algorithms in this paper. Paths are colored in some order defined later. At any stage of the coloring procedure, a path is called *colored* if it has been assigned a color, otherwise *uncolored*. In the algorithm, processing a node u means coloring the uncolored paths on u. We call a node u *processed* if the coloring process for u has been completed, otherwise *unprocessed*. Notice that before the coloring process for node u, some paths on u may have been colored due to the processing of other nodes. Node u is still called unprocessed if the coloring process for u has not been performed even all paths on u have been colored due to the processing of other nodes. Our algorithm processes the nodes of TR in the DFS (depth-first search) order proposed in [8]. For a node u, its *parent* is the node from which u is reached in the DFS order (see (b) of Fig. 1). A link is called *special* if it connects a processed node and an unprocessed node (see (b) of Fig. 1). There are either 0 or 2 special links in a ring in TR. A path on a special link is colored and only such a path has a possibility to intersect with an uncolored path. We assume that in Step 1, the nodes in the same ring are searched in the clockwise direction in the DFS order. Notice that a node of degree two always exists in a finite TR.

The steps of Algorithm A1 are given in the framework in Fig. 2. In Step 2, we first assign colors of W to the paths on link $(u_0, {u_0}^-)$ by the first-fit coloring. Next we assign the uncolored paths on link $(u_0, {u_0}^+)$ the colors of $W \setminus W_{u_0 u_0^-}$ by the first-fit coloring.

In Step 3, the parent of node u in the DFS order is node u^- in some ring which is called r_0. If u appears in $k+1$ rings, the other k rings are denoted by r_i, $1 \le i \le k$ (see (b) of Fig. 1). Let Q_0 be the set of paths on special links (u, u^-) or (w, w^-). In Step 3.1, we color P_0 using the colors of $W \setminus W_{Q_0}$ by the first-fit coloring.

In Step 3.2, we convert the path coloring problem to the edge-coloring problem of a multigraph G_u with rings r_i $(0 \le i \le k)$ as vertices and all paths on u as edges. By Proposition 1, a path on u is on either one ring or two rings. A path on u is called a *long path* if it is on two rings, otherwise a *short path* (see (b) of Fig. 1). To eliminate self-loops, we introduce a vertex s_i for every r_i in G_u. More specifically, G_u is defined as: $V(G_u) = \{r_i, s_i | 0 \le i \le k\}$, and

$$E(G_u) = \{(r_i, r_j, p) \mid p \text{ is a long path on } r_i \text{ and } r_j, 0 \le i < j \le k\} \cup \{(r_i, s_i, p) \mid p \text{ is a short path on } u \text{ and } r_i, 0 \le i \le k\},$$

where (x, y, p) denotes an edge between vertices x and y with label p. From Proposition 1, there is a one-to-one correspondence between the paths on u and the edges in G_u. Assume that a valid edge-coloring for G_u has been found and let

Procedure Framework(TR,P)
Input: A set P of paths in TR.
Output: A valid coloring from $W = \{\lambda_1, \lambda_2, \ldots\}$ to P.
begin
1. Fix a DFS (depth-first search) order, starting from a node (say u_0) of degree two, on the nodes of TR.
2. Process the starting node u_0.
3. Process the other nodes u in the DFS order.
   Let r_0 be the ring which contains u and the parent of u.
   3.1 Color the set P_0 of uncolored paths on u and r_0.
   3.2 Color the set P_1 of other uncolored paths on u.
end.

Fig. 2. A framework of algorithms for the WA problem on trees of rings.

$C_{G_u} = \{\mu_1, \mu_2, \ldots\}$ be the set of virtual colors used for the edge-coloring. We use the mapping $f_1 : C_{G_u} \to W$ defined below to color the corresponding paths on u. Let Q_1 be the set of colored paths on u before Step 3.2 and C_{Q_1} be the set of virtual colors assigned to the edges (x, y, p) of G_u with $p \in Q_1$. The mapping f_1 is defined as follows:

(1) For each $\mu_i \in C_{Q_1}$ assigned to edge (x, y, p) with $p \in Q_1, f_1(\mu_i) = \lambda_j$, where $\lambda_j \in W_{Q_1}$ is the color assigned to path p before Step 3.2.
(2) For each $\mu_i \in C_{G_u} \setminus C_{Q_1}, f_1$ maps μ_i to a $\lambda_j \in W \setminus W_{Q_1}$ with the smallest available index j such that $C_{G_u} \setminus C_{Q_1}$ is assigned distinct colors.

Since all paths of Q_1 are on ring r_0, edges (x, y, p) with $p \in Q_1$ are given distinct virtual colors in the edge-coloring of G_u. From this and the above definition, f_1 is a function from C_{G_u} to W which implies that for any two edges (in G_u), the corresponding paths are colored by the same real color if and only if these two edges are colored by the same virtual color. Also, f_1 does not change the colors of the paths which were colored before Step 3.2.

To apply the edge-coloring of G_u in Step 3.2 as shown above, it is required that Q_1 has been assigned distinct colors. In other words, no color has been given to more than one path in Q_1. A set of colored paths is called a *0-set* if the paths are assigned distinct colors. We say the 0-set condition is true on a ring if the set of paths on special links of the ring is a 0-set. As shown later, the 0-set condition is kept true for each ring of TR in Algorithm A1. This is critical in applying the edge-coloring of G_u in Step 3.2.

Algorithm A1 colors the paths step by step. In each step, there are a set of colored paths and a set of paths to be colored. The following lemma is useful to get the total number of colors from that used for coloring in each step.

Lemma 1. *Given a set Q of colored paths and a set R of uncolored paths, assume that at most w colors have been used for Q, and that a subset Q' of Q contains every colored path intersecting with a path of R. If an algorithm colors R such that the coloring for R and the previous coloring for Q' give a valid coloring for $Q' \cup R$ using at most w colors, then $Q \cup R$ can be colored with at most w colors.*

Proof. From the condition of the lemma, $|W_R \setminus W_{Q'}| = |W_R \cup W_{Q'}| - |W_{Q'}| \leq w - |W_{Q'}|$. Since each path of $Q \setminus Q'$ is edge-disjoint with any path of R, all colors of $W_Q \setminus W_{Q'}$ can be used as the colors for R. Therefore, $Q \cup R$ can be colored with at most w colors. $\square$

By the above lemma, if an algorithm colors R such that at most w colors are used for $Q' \cup R$ in every step, it solves the WA problem with at most w colors. In what follows, we only analyze the number of colors used in each step for $Q' \cup R$. Let Q_u be the set of colored paths and P_u be the set of paths to be colored when node u is being processed in Algorithm A1.

Theorem 1. *Algorithm A 1 solves the WA problem on TR with n nodes and degree $2(k+1)$ using at most $3L$ colors in $O(nkL(k+L))$ time.*

Proof. In Step 2, since there are at most L paths on any link of TR, there are at most $2L$ paths on u_0. Therefore, the paths on u_0 can be colored with at most $2L$ colors. Since each path is given a distinct color, the 0-set condition is true for every ring of TR after this step. We now show that Algorithm A1 colors $Q_u \cup P_u$ using at most $3L$ colors for every node u in TR.

In Step 3, assume that Q_u has been colored with at most $3L$ colors and the 0-set condition is true for every ring of TR. In Step 3.1, $|P_0| \leq L$ because each path in P_0 is on link (u, u^+). Therefore, P_0 can be colored with at most L colors. Since Q_0 defined for Step 3.1 of Algorithm A1 contains every colored path intersecting with a path of P_0 and $|Q_0| \leq 2L$, $Q_0 \cup P_0$ (thus $Q_u \cup P_0$ by Lemma 1) can be colored with most $3L$ colors. Since $Q_0 \cup P_0$ is assigned distinct colors, after Step 3.1 the 0-set condition is true for r_0 and Q_1 defined for Step 3.2 of Algorithm A1 is a 0-set. The latter is critical in Step 3.2.

In Step 3.2, Q_1 contains every colored path intersecting with a path of P_1. By the edge-coloring of G_u, the definition of mapping f_1, and the 0-set condition of Q_1, the set of paths on ring r_i and node u is assigned distinct colors. Also f_1 does not change the color of any path in Q_1. Therefore, f_1 colors P_1 such that the colorings for P_1 and Q_1 give a valid coloring for $Q_1 \cup P_1$.

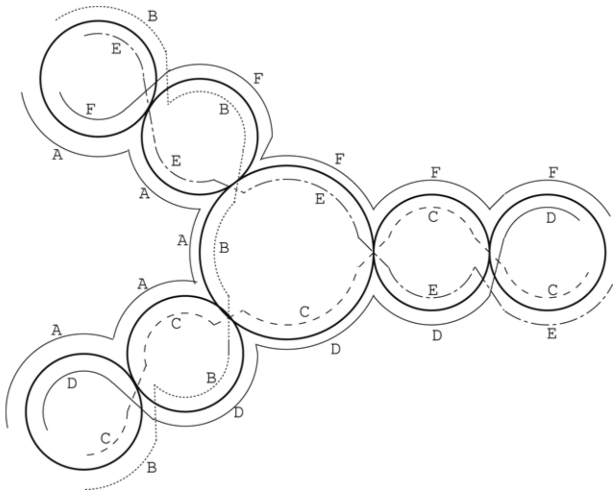

Fig. 3. An instance for the $3L$ lower bound.

The number of colors used for $Q_1 \cup P_1$ is $|C_{G_u}|$. Since there are at most L paths on any link of TR, there are at most $2L$ paths on node u and any ring r_i. Therefore, $\Delta(G_u) \leq 2L$. By Proposition 2, a valid edge-coloring of G_u can be found using $|C_{G_u}| \leq 3L$ colors. Thus, at most $3L$ colors are used for $Q_1 \cup P_1$, implying at most $3L$ colors for $Q_u \cup P_u$. Since the set of paths on u and any ring r_i is assigned distinct colors, the 0-set condition holds for every ring.

Summarizing the above, the algorithm solves the WA problem on TR using at most $3L$ colors. The edge-coloring of multigraph G_u is the dominant part in Algorithm A1 for the time complexity. Since $\Delta(G_u) \leq 2L$, $|V(G_u)| \leq 2(k+1)$, and $|E(G_u)| = O(kL)$, by Proposition 2, the edge-coloring of G_u takes $O(kL(k+L))$ time. Since Algorithm A1 executes Steps 3.1 and 3.2 $O(n)$ times, the time complexity of the algorithm is $O(nkL(k+L))$. $\square$

It is known that there are instances which require $3L$ colors for the WA problem on trees of rings [4]. An example of such instances is as follows: Let $P = A \cup B \cup C \cup D \cup E \cup F$ be the set of paths, with each subset having $L/2$ (L is even) paths, as shown in Fig. 3. The maximum number of paths on any link in the tree of rings is L. Any two paths in P must be assigned different colors since they intersect with each other. There are a total of $3L$ paths in P, thus $3L$ colors are needed. This lower bound shows that in the worst case one cannot do better than $3L$ even for trees of rings with node degree four. Algorithm A1 achieves the $3L$ tight upper bound for trees of rings with arbitrary degrees. Since L is a lower bound on the number of wavelengths for any optimal solution, Algorithm A1 achieves an approximation ratio of 3 for the WA problem on TR with an arbitrary degree. The algorithm can be used to obtain a 3-approximation algorithm for the RWA problem on trees of rings as following. First, for a given set of connection requests, a path for each request can be found efficiently such that L is minimized [8]. Then, the set of found paths is colored by Algorithm A1 using at most $3L$ colors. Since the load L is optimal, it is also a lower bound for the original RWA problem. In this way, the approximation ratio of 3 is achieved without using the cut-one-link approach.

4. Preparation for improvement

In Algorithm A1, the 0-set condition is kept for every ring for edge-coloring G_u in Step 3.2 in a straightforward way. One observation is that the 0-set condition may be too strict for solving the WA problem on TR since two paths on special links of a ring can have the same color if they are edge-disjoint. Better approximation ratios may be achieved if the 0-set condition is relaxed. Another observation is that we may use less colors for the edge-coloring of multigraph G_u if a more advanced algorithm like that in [15] is used. In this section, we first give two schemes for coloring paths on trees of rings with the 0-set condition relaxed. The path coloring schemes make more efficient use of colors. Then we show some properties of multigraph G_u related to its edge-coloring. The path coloring schemes and properties of G_u will be used in the following sections to get algorithms with better approximation ratios.

4.1. Efficient path coloring schemes

We first introduce the notion of β-set which is an extension of 0-set. A color for a set of colored paths is called a *multi-color* if the color has been assigned to two paths in the set. For an integer $\beta \geq 0$, a set of colored paths is called a *β-set* if each color is assigned to at most two paths, the paths with the same color are edge-disjoint, and the number of multi-colors for the path set is at most β. We say the β-set condition is true on a ring if the set of paths on special links of the ring is a β-set. For any given integer β with $0 \leq \beta \leq L$, the schemes given below use as few colors as possible to keep the β-set condition for every ring.

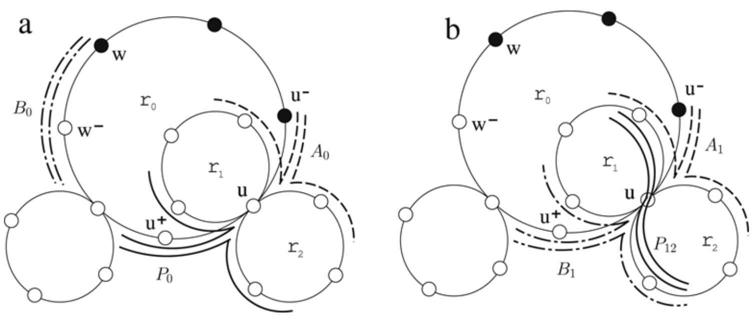

Fig. 4. The sets of paths related to Schemes S31 and S32.

Recall that P_0 and P_1 are the sets of paths to be colored in Step 3.1 and Step 3.2 of the framework in Fig. 2, respectively. We give a scheme for coloring P_0 and a scheme for coloring a subset of P_1. The scheme for P_0, called S31, works as follows. Assume that the β-set condition is true for every ring before P_0 is colored. Recall that Q_0 is the set of paths on special links (u, u^-) or (w, w^-). Let $W^m_{Q_0} \subseteq W_{Q_0}$ be the set of multi-colors for Q_0. Then from the β-set condition, $|W^m_{Q_0}| \leq \beta$. Define A_0 (resp. B_0) to be the set of paths on link (u, u^-) (resp. on (w, w^-)), each of which has a color in $W_{Q_0} \setminus W^m_{Q_0}$ (see (a) of Fig. 4). Then $|A_0| + |W^m_{Q_0}| \leq L$, $|B_0| + |W^m_{Q_0}| \leq L$, and $A_0 \cup B_0$ is assigned distinct colors. We construct a graph G_0 with

$$V(G_0) = P_0 \cup A_0 \quad \text{and} \quad E(G_0) = \{(p, q) \mid p \text{ and } q \text{ are edge-disjoint}\}.$$

We find a maximum matching M_0 of G_0. Notice that G_0 is bipartite and for each pair $(p, q) \in M_0$, $p \in P_0$ and $q \in A_0$. We select $\min\{|M_0|, \beta - |W^m_{Q_0}|\}$ pairs from M_0. For each selected pair (p, q), assign the color of $q \in A_0$ to p. We assign the remaining paths of P_0 the colors of $W \setminus W_{Q_0}$ by the first-fit coloring. As shown later, the β-set condition is true for every ring after P_0 is colored.

The second scheme, called S32, is used to color a subset of P_1. More specifically, S32 is used to color the long paths on rings r_i and r_j ($i, j \neq 0$, $i \neq j$) subject to the condition that every colored path on r_i or r_j is also on r_0 when S32 is called. Without loss of generality, we assume that $r_i = r_1$ and $r_j = r_2$ for simplicity. Let $P_{12} \subseteq P_1$ be the set of long paths on rings r_1 and r_2 (see (b) of Fig. 4). Recall that Q_1 is the set of colored paths on u before Step 3.2. Then every path of Q_1 is on r_0. Let $Q'_1 \subseteq Q_1$ be the set of colored long paths on links (u, u^-) or (u, u^+) and on rings r_1 or r_2. We define $W^m_{Q'_1} \subseteq W_{Q'_1}$ to be the set of multi-colors for the set Q'_1. From the β-set condition, $|W^m_{Q'_1}| \leq \beta$. Define A_1 (resp. B_1) to be the set of long paths on link (u, u^-) (resp. on (u, u^+)) and on rings r_1 or r_2, each of which has a color in $W_{Q'_1} \setminus W^m_{Q'_1}$ (see (b) of Fig. 4). Then $|A_1| + |W^m_{Q'_1}| \leq L$, $|B_1| + |W^m_{Q'_1}| \leq L$, and $A_1 \cup B_1$ is assigned distinct colors. We construct a graph G_1 with

$$V(G_1) = P_{12} \cup A_1 \quad \text{and} \quad E(G_1) = \{(p, q) \mid p \text{ and } q \text{ are edge-disjoint}\}.$$

We find a maximum matching M_1 of G_1. For each pair $(p, q) \in M_1$, either $p \in P_{12}$ and $q \in A_1$ or $p, q \in P_{12}$. We select $\min\{|M_1|, \beta - |W^m_{Q'_1}|\}$ pairs from M_1. For each selected pair (p, q) with $q \in A_1$, assign the color of q to p. For each selected pair (p, q) with $p, q \in P_{12}$, assign the pair a distinct color from $W \setminus W_{Q'_1}$ by the first-fit coloring (p and q share the same color, but different pairs are given different colors). Let W' be the set of colors assigned to the selected pairs (p, q) with $p, q \in P_{12}$. We assign each of the remaining paths of P_{12} a color from $W \setminus (W_{Q'_1} \cup W')$ by the first-fit coloring such that the set of the remaining paths of P_{12} is assigned distinct colors.

Let OPT_0 (resp. OPT_1) be the number of colors required to color $P_0 \cup A_0$ (resp. $P_{12} \cup A_1$) in an optimal solution.

Lemma 2. *$OPT_0 = |P_0| + |A_0| - |M_0|$ and $OPT_1 = |P_{12}| + |A_1| - |M_1|$.*

Proof. It is easy to see that $|P_0| + |A_0| - |M_0|$ colors are sufficient for $P_0 \cup A_0$. We prove that $|P_0| + |A_0| - |M_0|$ colors are also necessary. Notice that each color is used by at most two paths in $P_0 \cup A_0$. Assume to the contrary that there is a valid coloring which uses $OPT' = k_1 + k_2 < |P_0| + |A| - |M_0|$ colors, where each of the k_1 colors is used by one path and each of the k_2 colors is used by two paths. Since M_0 is a maximum matching, $k_2 \leq |M_0|$. The total number of paths colored by OPT' is

$$|P_0| + |A_0| = k_1 + 2k_2 \leq k_1 + k_2 + |M_0| < |P_0| + |A_0| - |M_0| + |M_0| = |P_0| + |A_0|,$$

a contradiction.

The proof for $OPT_1 = |P_{12}| + |A_1| - |M_1|$ is similar, noting that a color is used for at most two paths in $P_{12} \cup A_1$. □

Lemma 3. *Scheme S31 colors P_0 such that the colorings for Q_0 and P_0 give a valid coloring for $Q_0 \cup P_0$ using at least $|Q_0 \cup P_0| - \beta$ and at most $\min\{|Q_0 \cup P_0|, \max\{|Q_0 \cup P_0| - \beta, OPT_0 + L\}\}$ colors. Furthermore, $Q_0 \cup P_0$ is a β-set.*

Proof. Clearly, at most $|Q_0 \cup P_0|$ colors are used, paths with the same color are edge-disjoint, and a color is used for at most two paths in $Q_0 \cup P_0$. If $|M_0| > \beta - |W^m_{Q_0}|$ then there are exactly β multi-colors for $Q_0 \cup P_0$ and $|Q_0 \cup P_0| - \beta$ colors are used. Otherwise, there are at most β multi-colors for $Q_0 \cup P_0$ and at least $|Q_0 \cup P_0| - \beta$ colors are used. In the latter case, the number of colors used is $|A_0| + |B_0| + |W^m_{Q_0}| + |P_0| - |M_0|$. From Lemma 2 and $|B_0| + |W^m_{Q_0}| \leq L$, at most $OPT_0 + L$ colors are used. □

Lemma 4. *Scheme S32 colors P_{12} such that the colorings for Q_1' and P_{12} give a valid coloring for $Q_1' \cup P_{12}$ using at least $|Q_1' \cup P_{12}| - \beta$ and at most $\min\{|Q_1' \cup P_{12}|, \max\{|Q_1' \cup P_{12}| - \beta, OPT_1 + L\}\}$ colors. Furthermore, $Q_1' \cup P_{12}$ is a β-set.*

Proof. Clearly, at most $|Q_1' \cup P_{12}|$ colors are used, paths with the same color are edge-disjoint, and each color is used for at most two paths in $Q_1' \cup P_{12}$. If $|M_1| > \beta - |W^m_{Q_1'}|$ then there are exactly β multi-colors for $Q_1' \cup P_{12}$ and $|Q_1' \cup P_{12}| - \beta$ colors are used. Otherwise, there are at most β multi-colors for $Q_1' \cup P_{12}$ and at least $|Q_1' \cup P_{12}| - \beta$ colors are used. In the latter case, the number of colors used is $|A_1| + |B_1| + |W^m_{Q_1'}| + |P_{12}| - |M_1|$. By Lemma 2 and $|B_1| + |W^m_{Q_1'}| \le L$, at most $OPT_1 + L$ colors are used. □

4.2. Edge-coloring of multigraphs

For a (multi)graph G, $l(G)$ defined in Proposition 3 is a lower bound on the number of colors for the edge-coloring of G. The multigraph G_u constructed in Step 3.2 of Algorithm A1 has maximum degree $2L$, and $l(G_u)$ can be as large as $3L$. Thus, a direct application of more advanced edge-coloring algorithms (such as that of [15]) in Step 3.2 of Algorithm A1 cannot improve the approximation ratio. In this subsection, we show some properties of G_u when $l(G_u)$ is large. These properties, Schemes S31 and S32, and the application of a more advanced edge-coloring algorithm in Step 3.2 will be used to improve the approximation ratio of Algorithm A1.

Lemma 5. *For any subgraph H of G_u, if $L(H) > \lceil 2.5L \rceil$ then $|V(H)| = 3$.*

Proof. If $L(H) > \lceil 2.5L \rceil$, then clearly $|V(H)| \ge 3$. Therefore, it suffices to show that $L(H) \le \lceil 2.5L \rceil$ if $|V(H)| > 3$. Consider two cases. If $|V(H)| = 2j$ $(j \ge 2)$,

$$L(H) = \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil = \left\lceil \frac{|E(H)|}{j} \right\rceil \le \left\lceil \frac{\sum_{u \in V(H)} d(u)}{2j} \right\rceil \le \left\lceil \frac{|V(H)| \times 2L}{2j} \right\rceil = 2L.$$

If $|V(H)| = 2j + 1$ $(j \ge 2)$,

$$L(H) = \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil = \left\lceil \frac{|E(H)|}{j} \right\rceil \le \left\lceil \frac{\sum_{u \in V(H)} d(u)}{2j} \right\rceil \le \left\lceil \frac{|V(H)| \times 2L}{2j} \right\rceil = \left\lceil \left(1 + \frac{1}{2j}\right) \times 2L \right\rceil \le \lceil 2.5L \rceil. \quad \square$$

Lemma 6. *For subgraphs H_1 and H_2 of G_u with $L(H_1) > \lceil 2.5L \rceil$ and $L(H_2) > \lceil 2.5L \rceil$, $V(H_1) \cap V(H_2) = \emptyset$.*

Proof. To prove the lemma by contradiction, assume that $V(H_1) \cap V(H_2) \ne \emptyset$. By Lemma 5, both H_1 and H_2 have three vertices. There are two cases to consider: $|V(H_1) \cap V(H_2)| = 1$ and $|V(H_1) \cap V(H_2)| = 2$. For the first case, the total number of edges in $H_1 \cup H_2$ is

$$|E(H_1 \cup H_2)| = |E(H_1)| + |E(H_2)| > \lceil 2.5L \rceil + \lceil 2.5L \rceil \ge 5L.$$

However,

$$|E(H_1 \cup H_2)| \le \frac{\sum_{u \in (V(H_1) \cup V(H_2))} d(u)}{2} \le \frac{5 \times 2L}{2} = 5L,$$

a contradiction.

For the second case, assume that $V(H_1) \cap V(H_2) = \{r_a, r_b\}$. Let $m(r_a, r_b)$ be the number of multi-edges between r_a and r_b. The total number of edges in $H_1 \cup H_2$ is

$$|E(H_1 \cup H_2)| = |E(H_1)| + |E(H_2)| - m(r_a, r_b) > 5L - m(r_a, r_b).$$

However,

$$|E(H_1 \cup H_2)| \le d(r_a) + d(r_b) - m(r_a, r_b) \le 4L - m(r_a, r_b),$$

a contradiction. □

For multigraph G_u, let F_u be the graph obtained by contracting each subgraph H of G_u with $L(H) > \lceil 2.5L \rceil$ into a single vertex v_H. More precisely, let $V' = \cup_{H:L(H)>\lceil 2.5L \rceil} V(H)$ and $E' = \cup_{H:L(H)>\lceil 2.5L \rceil} E(H)$. Graph F_u is defined by

$$V(F_u) = \{v_H | L(H) > \lceil 2.5L \rceil\} \cup (V(G_u) \setminus V')$$

and

$$E(F_u) = E(G_u) \setminus E',$$

where for each edge in $E(G_u) \setminus E'$, if an end vertex of the edge is in $V(H)$, the end vertex is replaced by v_H in $E(F_u)$. From Lemma 6, $v_{H_1} \ne v_{H_2}$ for $H_1 \ne H_2$. We call F_u the *contracted graph* of G_u. Fig. 5 gives an example of G_u and F_u.

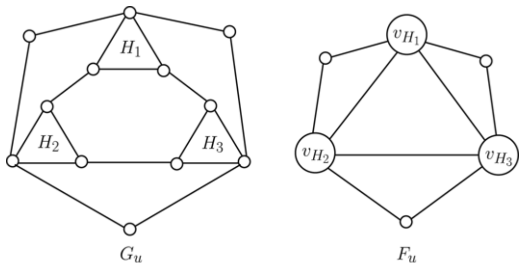

Fig. 5. A multigraph G_u and its contracted graph F_u.

Lemma 7. *$l(F_u) \le \lceil 2.5L \rceil$ and $d(v_H) < L$.*

Proof. The degree of v_H in graph F_u is

$$d(v_H) = \sum_{u \in V(H)} d(u) - 2 \times |E(H)| < |V(H)| \times 2L - 2 \times \lceil 2.5L \rceil \le 3 \times 2L - 5L = L.$$

After every subgraph H with $L(H) > \lceil 2.5L \rceil$ is contracted to a vertex v_H in F_u, from $d(v_H) < L$, any subgraph of F_u with three vertices including v_H has at most $\lfloor \frac{d(v_H)+2L+2L}{2} \rfloor < \lceil 2.5L \rceil$ edges. Therefore, by Lemma 5, $l(F_u) \le \lceil 2.5L \rceil$. $\square$

5. A 2.75-approximation algorithm

Applying the results of the previous section, we show a better approximation algorithm for the WA problem on TR with an arbitrary degree. By Proposition 3, the edge-coloring of G_u can be done with at most $\lceil 2.5L \rceil$ colors if $l(G_u) \le \lceil 2.5L \rceil$. On the other hand, if $l(G_u) > \lceil 2.5L \rceil$, we can contract G_u into F_u with $l(F_u) \le \lceil 2.5L \rceil$ (Lemma 7) and then apply the edge-coloring algorithm of [15] to F_u. Each contracted subgraph H has three vertices, corresponding to three rings containing node u, and the paths corresponding to the edges in H can be colored by Schemes S31 and S32, with a properly chosen integer β. For simplicity, in what follows, we sometimes refer to the edges in the multigraph G_u and the corresponding paths on node u of TR without distinguishing them, if there is no confusion.

Our algorithm, called A2, follows the framework in Fig. 2. Step 2 of A2 is the same as that in Algorithm A1. Step 3.1 uses Scheme S31. By Lemma 3, $|Q_0 \cup P_0| \le 3L$, and $OPT_0 \le OPT$, at most $\min\{3L, \max\{3L - \beta, OPT + L\}\}$ colors are used for $Q_0 \cup P_0$. In Step 3.2, we color P_1. Similar to Algorithm A1, we convert the path coloring problem to the edge-coloring problem of multigraph G_u, but we use the algorithm of [15] to solve the edge-coloring problem. There are two cases.

Case 1: $l(G_u) \le \lceil 2.5L \rceil$.

We apply the algorithm of [15] to G_u. Since Scheme S31 is used for Step 3.1, Q_1 is a β-set. If Scheme S31 is used with a $\beta > 0$, two paths of Q_1 may have been colored by the same multi-color from W_{Q_1}. To get a valid coloring from W to the paths of G_u, for each pair of paths $p, q \in Q_1$ with the same multi-color from W_{Q_1}, we re-assign a new virtual color $\mu_{pq} \notin C_{G_u}$ to p and q. Let C'_{G_u} (resp. C'_{Q_1}) be the set of virtual colors assigned to the paths of G_u (resp. Q_1) after the re-assignment. We map C'_{G_u} to W by mapping f_1 defined in Section 3 to get a valid coloring from W to the paths of G_u. More specifically, we perform the following:

(1) For each $\mu_i \in C'_{Q_1}$ assigned to edge (x, y, p) with $p \in Q_1, f_1(\mu_i) = \lambda_j$, where $\lambda_j \in W_{Q_1}$ is the color assigned to path p before Step 3.2.
(2) For each $\mu_i \in C'_{G_u} \setminus C'_{Q_1}, f_1$ maps μ_i to a $\lambda_j \in W \setminus W_{Q_1}$ with the smallest available index j such that $C'_{G_u} \setminus C'_{Q_1}$ is assigned distinct colors.

Since Q_1 is a β-set, $|C'_{G_u}| \le |C_{G_u}| + \beta$. Also notice that $\Delta(G_u) \le 2L$, and $\lfloor (11\Delta(G_u) + 8)/10 \rfloor \le \lfloor 2.2L + 0.8 \rfloor \le \lceil 2.5L \rceil$ for any positive integer L. From Proposition 3 and $l(G_u) \le \lceil 2.5L \rceil$, the valid coloring uses at most $\lceil 2.5L \rceil + \beta$ colors. This suggests a small β. However, the upper bound $\min\{3L, \max\{3L - \beta, OPT + L\}\}$ in Step 3.1 suggests a large β. To minimize $\max\{\lceil 2.5L \rceil + \beta, 3L - \beta\}$, we choose $\beta = \lfloor 0.25L \rfloor$ for Scheme S31 in Step 3.1. Notice that $\lceil 2.5L \rceil + \lfloor 0.25L \rfloor \le 3L - \lfloor 0.25L \rfloor = \lceil 2.75L \rceil$.

Case 2: $l(G_u) > \lceil 2.5L \rceil$.

From Lemma 5, there is at least one subgraph H of G_u with $L(H) > \lceil 2.5L \rceil$ and $|V(H)| = 3$. There are two subcases depending on whether ring r_0 is a vertex of some H or not.

Case 2.1: Ring r_0 is not a vertex of any subgraph H of G_u with $L(H) > \lceil 2.5L \rceil$.

We contract G_u to F_u. From Lemma 7, $l(F_u) \le \lceil 2.5L \rceil$. We then apply the algorithm of [15] to F_u and get a valid coloring for the paths corresponding to the edges of F_u by the mapping f_1 as we did in Case 1. After this we color the paths corresponding to the edges in each contracted subgraph H by virtual colors of $C_H = \{\nu_1, \nu_2, \ldots\}$, using Scheme S32 as a subroutine (the details will be given shortly). Notice that some paths between ring r_0 and a ring of H may have been colored by multi-colors.

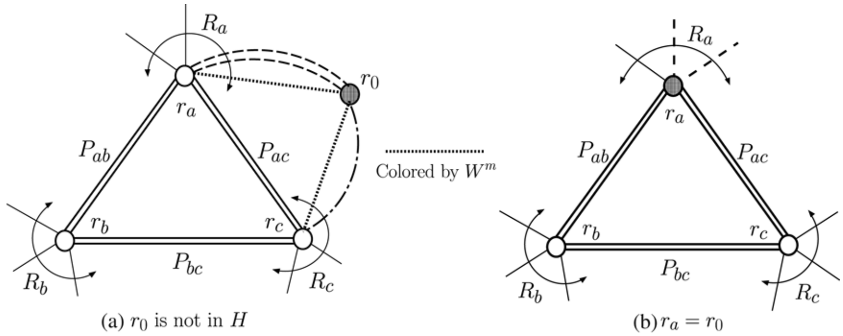

Fig. 6. Paths in and incident to subgraphs H with $L(H) > \lceil 2.5L \rceil$.

Because those multi-colors are also multi-colors for the paths on a ring of H, we need to subtract the number of those multi-colors from $\lfloor 0.25L \rfloor$ to get β for Scheme S32 to keep the $\lfloor 0.25L \rfloor$-set condition for each ring. We need some more definitions to formally define the β for Scheme S32.

Assume $V(H) = \{r_a, r_b, r_c\}$. We use P_{ij} ($i, j = a, b, c$; $i \neq j$) for the set of long paths in H on r_i and r_j, and use R_i ($i = a, b, c$) for the set of paths not in H but on r_i (see (a) of Fig. 6). Notice that $R_a \cup R_b \cup R_c$ has been colored and contains every colored path intersecting with a path of $P_{ab} \cup P_{ac} \cup P_{bc}$. Let $Q' = R_a \cup R_b \cup R_c$ and $W^m_{Q'}$ be the set of multi-colors for Q'. Since paths with a color from $W^m_{Q'}$ are on r_0, from the $\lfloor 0.25L \rfloor$-set condition on r_0, $|W^m_{Q'}| \leq \lfloor 0.25L \rfloor$.

For any two paths p and q with a multi-color $\lambda_m \in W^m_{Q'}$, there are two cases. Case (i), p and q are on r_0 and a single ring of H (say r_a, the dashed edges in (a) of Fig. 6). Case (ii), p and q are on r_0 and two rings of H (say r_a and r_c, the dotted edges in (a) of Fig. 6). In Case (i), λ_m is a multi-color for the ring of H (say r_a). In Case (ii), λ_m is not a multi-color for any ring of H. Let $W^m = \{\lambda_m | \lambda_m \in W^m_{Q'}$ is used in Case (ii)$\}$. Then at most $|W^m_{Q'}| - |W^m|$ colors of $W^m_{Q'}$ are multi-colors for each ring of H. From this, we take $\beta = \lfloor 0.25L \rfloor - |W^m_{Q'}| + |W^m|$ for applying Scheme S32 as a subroutine. To color $P_{ab} \cup P_{ac} \cup P_{bc}$ by virtual colors from C_H, we first assign $P_{ab} \cup P_{ac}$ distinct virtual colors from C_H. After this, a path colored with a virtual color from C_H on r_b or r_c must be in $P_{ac} \cup P_{ab}$ and thus is on r_a. Subject to this condition, we color the paths of P_{bc} with virtual colors from C_H using Scheme S32 with $\beta = \lfloor 0.25L \rfloor - |W^m_{Q'}| + |W^m|$, r_a, r_b, r_c corresponding to r_0, r_1, r_2 in the description of S32 in Section 4.1, respectively, $P_{ab} \cup P_{ac}$ corresponding to Q'_1, and P_{bc} corresponding to P_{12}.

After $P_{ab} \cup P_{ac} \cup P_{bc}$ is colored by virtual colors of C_H, we map the virtual colors to the colors of W. In the mapping, we try to use the colors of W_{R_a} to paths in P_{bc}. Similarly, we try to use the colors of W_{R_b} (resp. W_{R_c}) to paths in P_{ac} (resp. P_{ab}). Notice that the colors of W^m are not used in the mapping to keep the $\lfloor 0.25L \rfloor$-set condition on each ring. Let $C_{P_{ij}}$ ($i, j = a, b, c$; $i \neq j$) be the set of virtual colors for P_{ij}. We define mapping $f_2 : C_H \to W$ to color the paths in H as follows.

(1) Select $|W_{R_a} \setminus W^m|$ virtual colors from $C_{P_{bc}} \setminus (C_{P_{ab}} \cup C_{P_{ac}})$ arbitrarily, and f_2 maps each selected color ν_i to a $\lambda_j \in W_{R_a} \setminus W^m$ such that the selected virtual colors are assigned distinct real colors.

Similarly, select $|W_{R_b} \setminus W^m|$ (resp. $|W_{R_c} \setminus W^m|$) virtual colors from $C_{P_{ac}} \setminus (C_{P_{ab}} \cup C_{P_{bc}})$ (resp. $C_{P_{ab}} \setminus (C_{P_{ac}} \cup C_{P_{bc}})$), and f_2 maps each selected virtual color ν_i to a $\lambda_j \in W_{R_b} \setminus W^m$ (resp. $\lambda_j \in W_{R_c} \setminus W^m$) such that the selected virtual colors are assigned distinct real colors.

Let C_s be the set of selected virtual colors.

(2) For each $\nu_i \in C_H \setminus C_s$, f_2 maps ν_i to a $\lambda_j \in W \setminus (W_{R_a} \cup W_{R_b} \cup W_{R_c})$ with the smallest available index j such that $C_H \setminus C_s$ is assigned distinct colors.

The intuition of Step (1) of f_2 is to use as many colors of $W_{R_a} \cup W_{R_b} \cup W_{R_c}$ for $E(H)$ as possible. It is shown later that $|C_{P_{bc}} \setminus (C_{P_{ab}} \cup C_{P_{ac}})| \geq |W_{R_a} \setminus W^m|$, $|C_{P_{ac}} \setminus (C_{P_{ab}} \cup C_{P_{bc}})| \geq |W_{R_b} \setminus W^m|$, and $|C_{P_{ab}} \setminus (C_{P_{ac}} \cup C_{P_{bc}})| \geq |W_{R_c} \setminus W^m|$. This implies that Step (1) of f_2 can be done.

Case 2.2: Ring r_0 is a vertex of some H with $L(H) > \lceil 2.5L \rceil$.

Assume $V(H) = \{r_a = r_0, r_b, r_c\}$ (see (b) of Fig. 6). Notice that $P_{ab} \cup P_{ac} \cup R_a$ has been colored in Step 3.1 and every colored path on r_b or r_c is also on r_a. We color P_{bc} by Scheme S32 with $\beta = \lfloor 0.25L \rfloor$, r_a, r_b, r_c corresponding to r_0, r_1, r_2 in the description of S32 in Section 4.1, respectively, $P_{ab} \cup P_{ac}$ corresponding to Q'_1, and P_{bc} corresponding to P_{12}. Next we color R_b (resp. R_c), trying to use the colors of P_{ac} (resp. P_{ab}). Note that a color used by P_{ac} may have already been assigned to a path in P_{ab} or P_{bc}, and thus cannot be assigned to R_b without violating the $\lfloor 0.25L \rfloor$-set condition for ring r_b (similar situation exists for the set R_c and the ring r_c). To keep the $\lfloor 0.25L \rfloor$-set condition for rings r_b and r_c and for the set $R_a \cup R_b \cup R_c$, we do not use multi-colors for R_b and R_c. More specifically, let $W^m_{Q'}$ be the set of multi-colors for $Q' = P_{ab} \cup P_{ac} \cup R_a$ and $W^m_{Q''}$ be the set of multi-colors for $Q'' = P_{ab} \cup P_{ac} \cup P_{bc}$. Let $W_{P_{ab}}$ (resp. $W_{P_{ac}}$) be the set of colors for the paths of P_{ab} (resp. P_{ac}). For each path $p \in R_b$ (resp. $p \in R_c$), assign p a color from $W_{P_{ac}} \setminus (W^m_{Q'} \cup W^m_{Q''})$ (resp. $W_{P_{ab}} \setminus (W^m_{Q'} \cup W^m_{Q''})$) such that R_b (resp. R_c) is assigned distinct colors. Finally, we contract the H to one vertex, called r'_0, in multigraph G_u to get another multigraph G'_u. In G'_u, $r'_0 \notin V(H)$ for any subgraph H of G'_u with $L(H) > \lceil 2.5L \rceil$ (by Lemma 7) and the set of paths incident to r'_0 is colored and satisfies the $\lfloor 0.25L \rfloor$-set condition (this will be shown in the proof). We solve the edge-coloring of G'_u as in previous cases to get a valid path coloring.

Theorem 2. *Algorithm A2 solves the WA problem on TR with n nodes and degree $2(k+1)$ using at most* $\min\{3L, \max\{\lceil 2.75L\rceil, OPT+\lfloor 1.25L\rfloor\}\}$ *colors in* $O(nkL(k+L^{1.5}))$ *time.*

Proof. We show that for every node u, Algorithm A2 colors $Q_u \cup P_u$ with at most $\min\{3L, \max\{\lceil 2.75L\rceil, OPT+\lfloor 1.25L\rfloor\}\}$ colors. In Step 2, we get a valid coloring for $Q_u \cup P_u$ and the $\lfloor 0.25L\rfloor$-set condition for every ring of TR with $2L$ colors. In Step 3.1, by Lemma 3, we get a valid coloring for $Q_0 \cup P_0$ and the $\lfloor 0.25L\rfloor$-set condition with at most $\min\{3L, \max\{\lceil 2.75L\rceil, OPT+L\}\}$ colors. In Step 3.2, for Case 1 of $l(G_u) \le \lceil 2.5L\rceil$, by the $\lfloor 0.25L\rfloor$-set condition, the paths corresponding to edges in G_u can be colored with at most $\lceil 2.75L\rceil$ colors. Thus, we can get a valid coloring for $Q_u \cup P_u$ with at most $\min\{3L, \max\{\lceil 2.75L\rceil, OPT+\lfloor 1.25L\rfloor\}\}$ colors, and the $\lfloor 0.25L\rfloor$-set condition holds after the coloring.

For Case 2 of $l(G_u) > \lceil 2.5L\rceil$, in Case 2.1, we contract G_u into F_u and solve the edge-coloring of F_u. The paths corresponding to the edges of F_u can be colored with at most $\lceil 2.75L\rceil$ colors, according to Lemma 7 and the $\lfloor 0.25L\rfloor$-set condition. For each subgraph H of G_u with $L(H) > \lceil 2.5L\rceil$, the paths in H are colored by virtual colors of C_H and f_2 maps the virtual colors to real colors of W. To see Step (1) of f_2 can be done, we show that $|C_{P_{bc}} \setminus (C_{P_{ab}} \cup C_{P_{ac}})| \ge |W_{R_a} \setminus W^m|$, i.e., the number of virtual colors used only by P_{bc} is greater than or equal to the number of real colors used only by R_a. By Lemma 4, $|C_H| \ge |E(H)| - \beta$, where $\beta = \lfloor 0.25L\rfloor - |W^m_{Q'}| + |W^m|$ (recall that $Q' = R_a \cup R_b \cup R_c$, $W^m_{Q'}$ is the set of multi-colors for Q', and W^m is the subset of $W^m_{Q'}$ such that the two paths with a color from W^m are incident to different vertices of H). Since $|E(H)| > \lceil 2.5L\rceil$,

$$|C_H| \ge |E(H)| - \beta > \lceil 2.25L\rceil + |W^m_{Q'}| - |W^m| \ge \lceil 2.25L\rceil.$$

Therefore,

$$\begin{aligned} |C_{P_{bc}} \setminus (C_{P_{ab}} \cup C_{P_{ac}})| &\ge |C_H| - (|C_{P_{ab}}| + |C_{P_{ac}}|) \\ &\ge |C_H| - (|P_{ab}| + |P_{ac}|) \\ &> \lceil 2.25L\rceil - (d(r_a) - |R_a|) \\ &\ge \lceil 0.25L\rceil + |W_{R_a}| \\ &\ge |W_{R_a} \setminus W^m|. \end{aligned}$$

Similarly, $|C_{P_{ac}} \setminus (C_{P_{ab}} \cup C_{P_{bc}})| \ge |W_{R_b} \setminus W^m|$ and $|C_{P_{ab}} \setminus (C_{P_{ac}} \cup C_{P_{bc}})| \ge |W_{R_c} \setminus W^m|$. Summarizing the above, f_2 gives a valid coloring for $R_a \cup R_b \cup R_c \cup P_{ab} \cup P_{ac} \cup P_{bc}$. In addition, all the colors of $R_a \cup R_b \cup R_c$, except those in W^m, are mapped to the virtual colors in $C(H)$. The following calculations will show that the total number of colors used by $R_a \cup R_b \cup R_c \cup E(H)$ is at most $|C(H)| + |W^m|$ after the mapping f_2.

Since $|P_{ab} \cup P_{ac}| \le 2L$, $P_{ab} \cup P_{ac}$ can be colored with at most $2L$ distinct colors. Notice that $|E(H)| = |P_{ab} \cup P_{ac} \cup P_{bc}|$. By Lemma 4, $|C_H| \le \max\{|P_{ab} \cup P_{ac} \cup P_{bc}| - \beta, OPT+L\} = \max\{|E(H)| - (\lfloor 0.25L\rfloor - |W^m_{Q'}| + |W^m|), OPT+L\}$. The number of real colors used for $R_a \cup R_b \cup R_c \cup P_{ab} \cup P_{ac} \cup P_{bc}$ is at most, noting that $|E(H)| \le \lfloor \frac{6L-|Q'|}{2}\rfloor \le 3L - |W^m_{Q'}|$ and $|W^m| \le \lfloor 0.25L\rfloor$,

$$\begin{aligned} &|W_{R_a} \cup W_{R_b} \cup W_{R_c}| + \max\{|E(H)| - (\lfloor 0.25L\rfloor - |W^m_{Q'}| + |W^m|), OPT+L\} - (|W_{R_a} \setminus W^m| + |W_{R_b} \setminus W^m| + |W_{R_c} \setminus W^m|) \\ &\quad \le \max\{|E(H)| - \lfloor 0.25L\rfloor + |W^m_{Q'}| - |W^m|, OPT+L\} + |W^m| \\ &\quad \le \max\{3L - |W^m_{Q'}| - \lfloor 0.25L\rfloor + |W^m_{Q'}| - |W^m|, OPT+L\} + |W^m| \\ &\quad = \max\{\lceil 2.75L\rceil, OPT+L+|W^m|\} \\ &\quad \le \max\{\lceil 2.75L\rceil, OPT+\lfloor 1.25L\rfloor\}. \end{aligned}$$

Now we show the $\lfloor 0.25L\rfloor$-set condition is true for every ring. Notice that the paths of $R_a \cup R_b \cup R_c$ are given distinct virtual colors of C_{G_u} in the edge-coloring of F_u because all these paths are incident to the same vertex v_H. Therefore, only the paths incident to r_0 may be colored by real multi-colors when a valid coloring from W to the paths of F_u is found. This implies that $W_{R_i} \cap W_{R_j} \subseteq W^m$ $(i, j = a, b, c, i \ne j)$. From this, sets $(W_{R_i} \setminus W^m)$ $(i = a, b, c)$ are pairwisely disjoint. Recall that $C_{P_{ab}}$ and $C_{P_{ac}}$ are the sets of virtual colors from C_H assigned to P_{ab} and P_{ac}, respectively. The mapping f_2 selects a subset of $C_{P_{ab}}$ (resp. $C_{P_{ac}}$), assigns the subset distinct real colors from $W_{R_c} \setminus W^m$ (resp. $W_{R_b} \setminus W^m$), and assigns the remaining colors of $C_{P_{ab}} \cup C_{P_{ac}}$ distinct real colors from $W \setminus (W_{R_a} \cup W_{R_b} \cup W_{R_c})$. From $(W_{R_b} \setminus W^m) \cap (W_{R_c} \setminus W^m) = \emptyset$ and the fact that the paths in $P_{ab} \cup P_{ac}$ are given distinct virtual colors of C_H, the mapping f_2 assigns the paths in $P_{ab} \cup P_{ac}$ distinct real colors not in $W_{R_a} \setminus W^m$. Therefore, the $\lfloor 0.25L\rfloor$-set condition is true for r_a. There are at most $\beta = \lfloor 0.25L\rfloor - |W^m_{Q'}| + |W^m|$ virtual multi-colors of C_H for P_{bc} and each of them is mapped to a distinct real color in $W_{R_a} \setminus W^m$ or $W \setminus (W_{R_a} \cup W_{R_b} \cup W_{R_c})$. Therefore, there are at most $|W^m_{Q'}| - |W^m| + \beta = \lfloor 0.25L\rfloor$ real multi-colors for the paths on each of rings r_b and r_c. That is, the $\lfloor 0.25L\rfloor$-set condition is true for every ring.

In Case 2.2, $Q' = R_a \cup P_{ab} \cup P_{ac}$ has been colored with at most $2L$ colors and is a $\lfloor 0.25L\rfloor$-set. In addition, $P_{ab} \cup P_{ac}$ contains every colored path intersecting with a path of P_{bc}. By Lemma 4 and $|E(H)| \le 3L$, $Q'' = P_{ab} \cup P_{ac} \cup P_{bc}$ is colored with at most $\min\{3L, \max\{\lceil 2.75L\rceil, OPT+L\}\}$ colors. From this and Lemma 1, $R_a \cup P_{ab} \cup P_{ac} \cup P_{bc}$ is colored with at most $\min\{3L, \max\{\lceil 2.75L\rceil, OPT+\lfloor 1.25L\rfloor\}\}$ colors. On the other hand, by Lemma 4, Q'' is colored with at least $|E(H)| - \lfloor 0.25L\rfloor$ colors. Since $|E(H)| > \lceil 2.5L\rceil$, $|W^m_{Q'}| \le \lfloor 0.25L\rfloor$, $|W^m_{Q''}| \le \lfloor 0.25L\rfloor$, $W^m_{Q''} \setminus W^m_{Q'} \subseteq W_{P_{bc}}$, and $|W_{P_{ab}} \cup W_{P_{ac}} \cup W_{P_{bc}}| = |E(H)| - |W^m_{Q''}|$, we have

$$\begin{aligned} |W_{P_{ab}} \setminus (W^m_{Q'} \cup W^m_{Q''})| &\ge |W_{P_{ab}} \setminus W_{P_{bc}}| - |W^m_{Q'}| \\ &\ge (|E(H)| - |W^m_{Q''}| - |W_{P_{ac}}| - |W_{P_{bc}}|) - |W^m_{Q'}| \end{aligned}$$

$$> \lceil 2.25L \rceil - (\delta(r_c) - |R_c|) - \lfloor 0.25L \rfloor$$
$$\geq |R_c|.$$

Similarly, $|W_{P_{ac}} \setminus (W^m_{Q'} \cup W^m_{Q''})| \geq |R_b|$. Therefore, R_c (resp. R_b) is assigned distinct colors of $W_{P_{ab}} \setminus (W^m_{Q'} \cup W^m_{Q''})$ (resp. $W_{P_{ac}} \setminus (W^m_{Q'} \cup W^m_{Q''})$), and $R_a \cup R_b \cup R_c \cup P_{ab} \cup P_{ac} \cup P_{bc}$ is colored with at most $\min\{3L, \max\{\lceil 2.75L \rceil, OPT + \lfloor 1.25L \rfloor\}\}$ colors. The $\lfloor 0.25L \rfloor$-set condition holds for $R_a \cup R_b \cup R_c$ and each of rings r_a, r_b, and r_c, because no multi-color is introduced when coloring R_b and R_c. By solving the edge-coloring of G'_u, we can get a valid coloring for $Q_u \cup P_u$ with at most $\min\{3L, \max\{\lceil 2.75L \rceil, OPT + \lfloor 1.25L \rfloor\}\}$ colors.

The edge-coloring of G_u takes $O(kL(k + L))$ time by Proposition 3. It takes $O(L^{2.5})$ time to color a subgraph H of G_u with $L(H) > \lceil 2.5L \rceil$ (since H has degree at most $2L$, the graph constructed in Scheme S32 has $O(L)$ vertices, and it takes $O(L^{2.5})$ time to find a maximum matching in such graph [13]). There can be $O(k)$ such subgraphs H in G_u. Therefore, it takes $O(kL(k + L) + kL^{2.5}) = O(kL(k + L^{1.5}))$ time in Steps 3.1 and 3.2. The algorithm executes these steps $O(n)$ times. The time complexity of the algorithm is $O(nkL(k + L^{1.5}))$. □

Since $L \leq OPT$, $\min\{3L, \max\{\lceil 2.75L \rceil, OPT + \lfloor 1.25L \rfloor\}\}/OPT \leq \lceil 2.75L \rceil / L$. Thus Algorithm A2 achieves an approximation ratio of 2.75 asymptotically.

It seems difficult to extend the approach used in Algorithm A2 to improve the approximation ratio of 2.75 for a tree of rings with arbitrary degrees. One possible direction is to lower the threshold value to some $T < \lceil 2.5L \rceil$ for subgraph H. However, this will introduce the following problems. First, a subgraph H may have five or more vertices. A new scheme for coloring H is needed. Second, after the contraction of H in G_u, the resulting graph F_u may still have $l(F_u) > T$. To apply the edge-coloring algorithm, we may need to contract F_u as well. A new mapping function for converting the virtual colors of edge-coloring to real colors is needed. It is difficult to solve either of the problems. Nevertheless, in the next section, we show that the approach of Algorithm A2 can be used to derive algorithms with improved approximation ratios for bounded degree trees of rings.

6. Algorithms for bounded degrees

The ideas for the 2.75-approximation algorithm can be used to design more efficient algorithms for the WA problem on trees of rings with bounded degrees. Actually, Schemes S31 and S32 shown in Section 4 imply a $3L$ and 2-approximation algorithm for the WA problem on trees of rings with degree at most six. We first give the algorithm explicitly and then describe an algorithm for degree eight.

6.1. Algorithm for degree six

The algorithm, called A3, follows the framework in Fig. 2. Step 2 of A3 is the same as that in Algorithm A1. Step 3.1 uses Scheme S31. In Step 3.2, we first use Scheme S32 to color the long paths in P_{12}. Then we color the short paths on r_1 and those on r_2. Let Q' be the set of all long paths on u and r_1. We assign the short paths on r_1 the colors of $W \setminus W_{Q'}$ by the first-fit coloring such that the set of short paths is assigned distinct colors. Let Q'' be the set of all long paths on u and r_2. We assign the short paths on r_2 the colors of $W \setminus W_{Q''}$ by the first-fit coloring such that the set of short paths is assigned distinct colors.

Theorem 3. *Algorithm A3 solves the WA problem on TR with n nodes and degree at most six using at most* $\min\{OPT + L, 3L\}$ *colors in* $O(nL^{2.5})$ *time.*

Proof. To prove the theorem, we take $\beta = L$. We show that Algorithm A3 colors $Q_u \cup P_u$ using at most $\min\{OPT + L, 3L\}$ colors for every node u in TR.

In Step 2, $Q_u = \emptyset$ and $|P_u| \leq 2L$. At most $2L \leq \min\{OPT + L, 3L\}$ colors are used for $Q_u \cup P_u$. Obviously, the L-set condition is true for every ring after Step 2. In Step 3.1, by Lemma 3 and $|Q_0 \cup P_0| \leq 3L$, at most $\min\{3L, \max\{3L - L, OPT_0 + L\}\}$ colors are used for $Q_0 \cup P_0$. Since $OPT_0 \leq OPT$ and $L \leq OPT$, $\max\{3L - L, OPT_0 + L\} \leq OPT + L$ and at most $\min\{OPT + L, 3L\}$ colors are used for $Q_u \cup P_0$. By Lemma 3, the L-set condition is true for every ring after Step 3.1.

In Step 3.2, recall that $Q'_1 \subseteq Q_1$ is the set of colored long paths on links (u, u^-) or (u, u^+). Each path of Q'_1 (resp. P_{12}) is on one (resp. two) of the four links incident to u in r_1 and r_2, implying $|Q'_1| + 2|P_{12}| \leq 4L$. From this and $|Q'_1| \leq 2L$, we have $|Q'_1 \cup P_{12}| \leq 3L$. Notice that Q'_1 contains every colored path intersecting with a path of P_{12}. By Lemma 4, $OPT_1 \leq OPT$, $L \leq OPT$, and $|Q'_1 \cup P_{12}| \leq 3L$, at most $\min\{OPT + L, 3L\}$ colors are used for $Q'_1 \cup P_{12}$. By Lemma 4, the L-set condition is true for every ring after the coloring of P_{12}.

Since there are at most $2L$ paths on the two links of r_1 (resp. r_2) that are incident to node u, all the paths on the two links in r_1 (resp. r_2), including all the short paths, can be colored with $2L$ colors. Obviously, the L-set condition is true for every ring after the short paths are colored. Thus, Algorithm A3 colors $Q_u \cup P_u$ with at most $\min\{OPT + L, 3L\}$ colors and keeps the L-set condition for every ring.

A tree of rings TR with n nodes has $O(n)$ links. There are at most $O(nL)$ paths in a tree of rings with load L. To reduce the time complexity, we first construct a conflict graph G_c whose vertex set is P and two vertices of G_c are adjacent if the corresponding paths of P intersect with each other in TR. The conflict graph can be constructed in $O(nL^2)$ time, assuming that

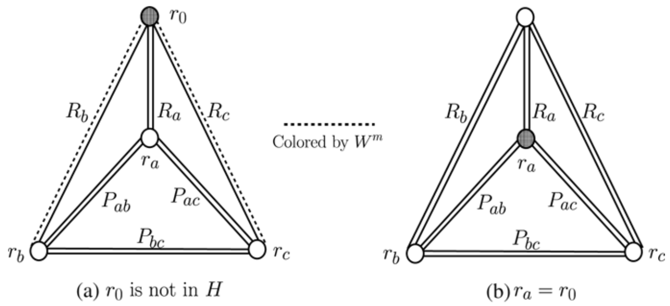

Fig. 7. Paths in and incident to the subgraph H with $L(H) > 2L$.

each path of P is given as a linked list of links of TR. The algorithm executes Steps 3.1 and 3.2 $O(n)$ times. The first-fit coloring takes $O(L^2)$ time to color L paths. It takes $O(L^2)$ time to construct a graph G_u of $O(L)$ vertices, by checking the conflict graph (there is an edge between two vertices of G_u if there is no edge between the two vertices in the conflict graph). It takes $O(L^{2.5})$ time to find a maximum matching of the graph [13]. Therefore, Steps 3.1 and 3.2 take $O(L^{2.5})$ time. The time complexity of the algorithm is $O(nL^{2.5})$. □

6.2. Algorithm for degree eight

The algorithm for degree eight, called A4, is similar to Algorithm A2, but uses a special scheme for the edge-coloring of multigraph G_u. Since the tree of rings considered has degree eight, G_u has at most four vertices r_i. Since the paths with an end vertex of s_i of G_u are short paths which can be easily colored with $2L$ colors after the long paths are colored, in what follows, we assume that G_u has only vertices r_i and edges corresponding to long paths. We first show an optimal edge-coloring algorithm for a multigraph with four vertices. We follow the notation used for Algorithm A2. Especially, for a subgraph H of multigraph G_u with $V(H) = \{r_a, r_b, r_c\}$, we use P_{ij} $(i, j = a, b, c; i \neq j)$ for the sets of long paths in H on r_i and r_j, and use R_i $(i = a, b, c)$ for the sets of paths not in H but on r_i.

Lemma 8. *An edge-coloring of multigraph G_u with four vertices can be done using at most* $\max\{\Delta(G_u), l(G_u)\}$ *colors in $O(|E(G_u)|)$ time.*

Proof. If $l(G_u) > \Delta(G_u)$, then there exists a subgraph H with three vertices which has $L(H) = l(G_u)$. To see this, assume that for any subgraph H with three vertices, $L(H) < l(G_u)$. The remaining vertex, which is not in H, has degree at most $\Delta(G_u)$. Then

$$l(G_u) \leq \left\lceil \frac{L(H) + \Delta(G_u)}{2} \right\rceil < l(G_u),$$

a contradiction. For the subgraph H with $L(H) = l(G_u)$, assume that $V(H) = \{r_a, r_b, r_c\}$ (see Fig. 7). We first color the edges $E(H)$ using $l(G_u)$ distinct colors. From $E(H) = l(G_u) > \Delta(G_u)$, we have

$$|P_{bc}| = |E(H)| - (d(r_a) - |R_a|) \geq |R_a|.$$

Thus, all edges of R_a can be colored by the colors used for P_{bc}, since each edge of R_a does not share a common vertex with any edge of P_{bc}. Similarly, all edges of R_b (resp. R_c) can be colored by the colors used for P_{ac} (resp. P_{ab}). Therefore, G_u can be edge-colored with at most $l(G_u)$ colors.

For the case of $l(G_u) \leq \Delta(G_u)$, assume that $d(r_0) = \Delta(G_u)$. We first color the edges incident to r_0 by $\Delta(G_u)$ distinct colors. Assume that the remaining vertices of G_u are r_a, r_b, and r_c (see (a) of Fig. 7). Let H be the subgraph with $V(H) = \{r_0, r_b, r_c\}$. Then $|E(H)| \leq l(G_u) \leq \Delta(G_u)$, and we have

$$|R_a| = d(r_0) - (|R_b| + |R_c|) = \Delta(G_u) - (|E(H)| - |P_{bc}|) \geq |P_{bc}|.$$

So, all edges of P_{bc} can be colored by the colors used for R_a. Similarly, all edges of P_{ab} (resp. P_{ac}) can be colored by the colors used for R_c (resp. R_b). Thus, G_u can be edge-colored with at most $\Delta(G_u)$ colors.

The algorithm first needs to find the larger number of $l(G_u)$ and $\Delta(G_u)$. This takes $O(|E(G_u)|)$ time. The coloring takes $O(|E(G_u)|)$ time. Thus, the time complexity of the algorithm is $O(|E(G_u)|)$. □

Algorithm A4 follows the framework of Fig. 2. Step 2 of A4 is the same as that in Algorithm A1. Step 3.1 uses Scheme S31 to color P_0 taking $\beta = \lfloor 0.5L \rfloor$. In Step 3.2 of A4, to color P_1, we convert the path coloring problem to the edge-coloring problem of multigraph G_u. Similar to Algorithm A2, there are two cases.

Case 1: $l(G_u) \leq 2L$.

In this case we edge-color G_u by the algorithm given in Lemma 8 using at most $2L$ virtual colors, re-assign virtual colors to the paths which have been assigned multi-colors of W before Step 3.2, and apply the mapping f_1 to color the paths of P_1, as we did in Case 1 of Step 3.2 of Algorithm A2.

Case 2: $l(G_u) > 2L$.

In this case, there is a subgraph H of G_u with $L(H) = l(G_u)$. There are two subcases.

Case 2.1: Ring r_0 is not a vertex in H.

We color the paths in H by virtual colors of $C_H = \{v_1, v_2, \ldots\}$. Assume that $V(H) = \{r_a, r_b, r_c\}$. Notice that $R_a \cup R_b \cup R_c$ has been colored. Let $W^m_{Q'}$ be the set of multi-colors for $Q' = R_a \cup R_b \cup R_c$. From the $\lfloor 0.5L\rfloor$-set condition, $|W^m_{Q'}| \le \lfloor 0.5L\rfloor$. Let W^m be the subset of $W^m_{Q'}$ such that the two paths with a color from W^m are incident to different vertices of H (see (a) of Fig. 7). Similar to Case 2.1 of Algorithm A2, we color $P_{ab} \cup P_{ac}$ by distinct virtual colors and P_{bc} by Scheme S32 with

$$\beta = \min\{\lfloor 0.5L\rfloor - |W^m_{Q'}| + |W^m|, |E(H)| - 2L\},$$

using virtual colors of C_H. Then we apply mapping f_2 in the same way as that in Algorithm A2, using as many as possible of the colors of W_{R_a}, W_{R_b}, and W_{R_c} to color P_{bc}, P_{ac}, and P_{ab}, respectively.

Case 2.2: Ring r_0 is a vertex of H.

Assume $V(H) = \{r_a = r_0, r_b, r_c\}$ (see (b) of Fig. 7). Notice that $R_a \cup P_{ab} \cup P_{ac}$ has been colored with at most $2L$ colors. Let $W^m_{Q'}$ be the set of multi-colors for $Q' = P_{ab} \cup P_{ac}$. Similar to Case 2.2 of Algorithm A2, we color P_{bc} by Scheme S32 with

$$\beta = \max\{|E(H)| - \lceil 2.5L\rceil, |W^m_{Q'}|\},$$

using colors of $W_{R_a} \setminus (W_{P_{ab}} \cup W_{P_{ac}})$ first and then colors of $W \setminus (W_{R_a} \cup W_{P_{ab}} \cup W_{P_{ac}})$. After this we assign R_b distinct colors from $W'_{P_{ac}} = W_{P_{ac}} \setminus (W_{R_a} \cup W_{P_{ab}} \cup W_{P_{bc}})$ first and then from $W \setminus (W_{R_a} \cup W_{P_{ab}} \cup W_{P_{ac}} \cup W_{P_{bc}})$ by the first-fit coloring such that R_b is assigned distinct colors. Similarly, we assign R_c distinct colors from $W'_{P_{ab}} = W_{P_{ab}} \setminus (W_{R_a} \cup W_{P_{ac}} \cup W_{P_{bc}})$ first and then from $W \setminus (W_{R_a} \cup W_{R_b} \cup W_{P_{ab}} \cup W_{P_{ac}} \cup W_{P_{bc}})$ by the first-fit coloring.

Theorem 4. *Algorithm A4 solves the WA problem on TR with n nodes and degree eight using at most* $\min\{3L, OPT + \lceil 1.5L\rceil\}$ *colors in* $O(nL^{2.5})$ *time.*

Proof. We prove that Algorithm A4 colors $Q_u \cup P_u$ with at most $\min\{3L, OPT + \lceil 1.5L\rceil\}$ colors for every node u. Similar to the proof for Algorithm A2, if $l(G_u) \le 2L$, we get a valid coloring for $Q_u \cup P_u$ with at most $\min\{3L, OPT + \lceil 1.5L\rceil\}$ colors and the $\lfloor 0.5L\rfloor$-set condition holds after the coloring.

Assume that $l(G_u) > 2L$ in Step 3.2. In Case 2.1, by Lemma 4, the paths in H are colored with at least $|E(H)| - \beta \ge 2L$ virtual colors, where $\beta = \min\{\lfloor 0.5L\rfloor - |W^m_{Q'}| + |W^m|, |E(H)| - 2L\}$, $Q' = R_a \cup R_b \cup R_c$, and $W^m_{Q'}$ is the set of multi-colors on Q'. Let $C_{P_{ij}}$ $(i, j = a, b, c;\ i \ne j)$ be the subset of virtual colors of C_H assigned to P_{ij}. We have

$$|C_{P_{bc}} \setminus (C_{P_{ab}} \cup C_{P_{ac}})| \ge |C_H| - |C_{P_{ab}}| - |C_{P_{ac}}| \ge 2L - (d(r_a) - |R_a|) \ge |W_{R_a}|.$$

Similarly, $|C_{P_{ac}} \setminus (C_{P_{ab}} \cup C_{P_{bc}})| \ge |W_{R_b}|$ and $|C_{P_{ab}} \setminus (C_{P_{ac}} \cup C_{P_{bc}})| \ge |W_{R_c}|$. Thus Step (1) of mapping f_2 can be done and we get a valid coloring for $R_a \cup R_b \cup R_c \cup E(H)$. The number of colors used is, noting $|E(H)| \le \lfloor \frac{6L-|Q'|}{2}\rfloor \le 3L - |W^m_{Q'}|$ and $|W^m| \le \lfloor 0.5L\rfloor$,

$$\begin{aligned}
&\max\{|E(H)| - \beta, OPT + L\} + |W^m| \\
&\quad \le \max\{|E(H)| - (\lfloor 0.5L\rfloor - |W^m_{Q'}| + |W^m|), |E(H)| - (|E(H)| - 2L), OPT + L\} + |W^m| \\
&\quad \le \max\{3L - |W^m_{Q'}| - \lfloor 0.5L\rfloor + |W^m_{Q'}| - |W^m|, OPT + L\} + |W^m| \\
&\quad = \max\{\lceil 2.5L\rceil, OPT + L + |W^m|\} \\
&\quad \le OPT + \lceil 1.5L\rceil.
\end{aligned}$$

$P_{ab} \cup P_{ac}$ are given distinct real colors. There are at most β virtual multi-colors of C_H for P_{bc} and each of them is mapped to a distinct real color in $W_{R_a} \setminus W^m$ or $W \setminus (W_{R_a} \cup W_{R_b} \cup W_{R_c})$. Notice that $W_{R_i} \cap W_{R_j} \subseteq W^m$ for $i, j = a, b, c;\ i \ne j$. Therefore, there are at most $|W^m_{Q'}| - |W^m| + \beta = \lfloor 0.5L\rfloor$ real multi-colors for the paths on each of rings r_a, r_b, and r_c. That is, the $\lfloor 0.5L\rfloor$-set condition is true for every ring.

In Case 2.2, P_{bc} is colored by Scheme $S32$ with $\beta = \max\{|E(H)| - \lceil 2.5L\rceil, |W^m_{Q'}|\}$, where $Q' = P_{ab} \cup P_{ac}$ and $W^m_{Q'}$ is the set of multi-colors on Q'. Since $|E(H)| \le 3L$, $\beta \le \lfloor 0.5L\rfloor$. Notice that Q' contains every colored path intersecting with a path of P_{bc}. By Lemmas 1 and 4, $R_a \cup E(H)$ can be colored with at most $\min\{3L, \max\{\lceil 2.5L\rceil, OPT + L\}\} \le \min\{3L, OPT + \lceil 1.5L\rceil\}$ colors. Recall that we choose $\beta = \max\{|E(H)| - \lceil 2.5L\rceil, |W^m_{Q'}|\}$ in the algorithm. Consider which of the two values β takes. Assume that $\beta = |E(H)| - \lceil 2.5L\rceil$. We show that R_b can be assigned distinct colors of $W'_{P_{ac}}$. Notice that

$$|W'_{P_{ac}}| \ge |W_{P_{ac}} \setminus W_{P_{bc}}| - |W_{P_{ac}} \cap (W_{R_a} \cup W_{P_{ab}})|.$$

By Lemma 4, at least $\lceil 2.5L\rceil$ colors are used for H. Since $W_{P_{ac}} \cap (W_{R_a} \cup W_{P_{ab}})$ is a subset of the multi-colors on $W_{R_a} \cup W_{P_{ab}} \cup W_{P_{ac}}$, from the $\lfloor 0.5L\rfloor$-set condition on ring r_a, $|W_{P_{ac}} \cap (W_{R_a} \cup W_{P_{ab}})| \le \lfloor 0.5L\rfloor$. Therefore,

$$\begin{aligned}
|W'_{P_{ac}}| &\ge (\lceil 2.5L\rceil - |W_{P_{ab}}| - |W_{P_{bc}}|) - \lfloor 0.5L\rfloor \\
&\ge \lceil 2.5L\rceil - (d(r_b) - |R_b|) - \lfloor 0.5L\rfloor \\
&\ge |R_b|.
\end{aligned}$$

From this, R_b can be assigned distinct colors of $W'_{P_{ac}}$. Similarly, R_c can be assigned distinct colors of $W'_{P_{ab}}$. Thus $R_a \cup R_b \cup R_c \cup E(H)$ can be colored with at most $\min\{3L, OPT + \lceil 1.5L \rceil\}$ colors, and from $\beta = |E(H)| - \lceil 2.5L \rceil \le \lfloor 0.5L \rfloor$ the $\lfloor 0.5L \rfloor$-set condition holds for $R_a \cup R_b \cup R_c$ and each of rings r_a, r_b, and r_c.

Assume that $\beta = |W^m_{Q'}|$. By the proof above, we can assume that $|W'_{P_{ac}}| < |R_b|$ or $|W'_{P_{ab}}| < |R_c|$. We further assume, without loss of generality, that $|W'_{P_{ac}}| < |R_b|$ and $|W'_{P_{ab}}| < |R_c|$ (the other two cases can be proved similarly). From $\beta = |W^m_{Q'}|$ and Scheme S32, P_{bc} is assigned distinct colors and $W_{P_{bc}} \cap (W_{P_{ab}} \cup W_{P_{ac}}) = \emptyset$. From $E(H) > 2L$ and $d(r_a) \le 2L$, $|P_{bc}| > |R_a|$ which implies that all colors of $W_{R_a} \setminus (W_{P_{ab}} \cup W_{P_{ac}})$ are used for P_{bc}. Therefore, $R_a \cup E(H)$ is colored with at most $E(H) - |W^m_{Q'}|$ colors. To color R_b and R_c,

$$(|R_b| - |W'_{P_{ac}}|) + (|R_c| - |W'_{P_{ab}}|)$$

additional colors are needed. Since $W_{P_{bc}} \cap (W_{P_{ab}} \cup W_{P_{ac}}) = \emptyset$,

$$\begin{aligned}(|R_b| - |W'_{P_{ac}}|) + (|R_c| - |W'_{P_{ab}}|) &= (|R_b| + |R_c|) - (|W_{P_{ac}} \setminus (W_{R_a} \cup W_{P_{ab}})| + |W_{P_{ab}} \setminus (W_{R_a} \cup W_{P_{ac}})|) \\ &\le (|R_b| + |R_c|) - (|W_{P_{ac}}| + |W_{P_{ab}}| - |W_{P_{ac}} \cap W_{R_a}| - |W_{P_{ab}} \cap W_{R_a}| - 2|W_{P_{ab}} \cap W_{P_{ac}}|).\end{aligned}$$

Since $(W_{P_{ac}} \cap W_{R_a}) \cup (W_{P_{ab}} \cap W_{R_a}) \cup (W_{P_{ab}} \cap W_{P_{ac}})$ is a subset of the multi-colors on $R_a \cup P_{ab} \cup P_{ac}$, from the $\lfloor 0.5L \rfloor$-set condition on ring r_a, $|W_{P_{ac}} \cap W_{R_a}| + |W_{P_{ac}} \cap W_{P_{ab}}| + |W_{P_{ab}} \cap W_{R_a}| \le \lfloor 0.5L \rfloor$ (noting that $W_{R_a} \cap W_{P_{ac}} \cap W_{P_{ab}} = \emptyset$). Since $W^m_{Q'}$ is the set of multi-colors on $Q' = P_{ab} \cup P_{ac}$,

$$|W_{P_{ac}}| + |W_{P_{ab}}| - |W_{P_{ab}} \cap W_{P_{ac}}| = |P_{ac}| + |P_{ab}| - |W^m_{Q'}|.$$

Also notice that $|P_{ac}| = |E(H)| - d(r_b) + |R_b|$ and $|P_{ab}| = |E(H)| - d(r_c) + |R_c|$. Summarizing the above,

$$\begin{aligned}(|R_b| - |W'_{P_{ac}}|) + (|R_c| - |W'_{P_{ab}}|) &\le |R_b| + |R_c| - (|P_{ac}| + |P_{ab}| - |W^m_{Q'}| - \lfloor 0.5L \rfloor) \\ &= d(r_b) + d(r_c) + |W^m_{Q'}| + \lfloor 0.5L \rfloor - 2|E(H)|.\end{aligned}$$

Since at most $|E(H)| - |W^m_{Q'}|$ colors are used for $R_a \cup E(H)$, $d(r_b)$, $d(r_c) \le 2L$, and $|E(H)| > 2L$, the total number of colors used for $R_a \cup R_b \cup R_c \cup E(H)$ is bounded by

$$d(r_b) + d(r_c) + |W^m_{Q'}| + \lfloor 0.5L \rfloor - 2|E(H)| + |E(H)| - |W^m_{Q'}| \le \lceil 2.5L \rceil.$$

Obviously, the $\lfloor 0.5L \rfloor$-set condition holds for $R_a \cup R_b \cup R_c$ and each of rings r_a, r_b, and r_c.

The edge-coloring of G_u takes $O(L)$ time by Lemma 8. It takes $O(L^{2.5})$ time to color the subgraph H of G_u with $L(H) > 2L$. The first-fit coloring takes $O(L^2)$ time to color L paths. Therefore, it takes $O(L^{2.5})$ time in Steps 3.1 and 3.2. The algorithm executes these steps $O(n)$ times. The time complexity of the algorithm is $O(nL^{2.5})$. □

It seems difficult to generalize the approach of Algorithm A4 for the WA problem on trees of rings with larger constant degrees, although a similar but more complicated analysis shows that the 2.5 approximation ratio is achievable for degree at most 10.

7. Conclusion

We gave a $3L$ and (asymptotic) 2.75-approximation algorithm for the WA problem on trees of rings with arbitrary degrees. The $3L$ upper bound is tight. We also presented a $3L$ and 2-approximation (resp. 2.5-approximation) algorithm for the WA problem on trees of rings with degree at most six (resp. eight). An interesting problem is to improve the 2.75-approximation ratio. A possible approach is to color the edges of multigraph G_u, allowing two edges with a common vertex in a given subset of edges sharing the same color. Another direction for the future work is to find better algorithms for the WA problem on trees of rings with constant degrees. Our results imply a 3-approximation algorithm for the RWA problem on a tree of rings. To the best of our knowledge, this is the first 3-approximation algorithm not based on the cut-one-link strategy. Our $3L$ algorithm also implies a $6L$ algorithm for the WA problem on directed trees of rings with two directed links, one in each direction, between a pair of adjacent nodes. It would be interesting to improve the approximation ratio for the WA problem on directed trees of rings.

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